

Badly approximable vectors in affine subspaces: Jarník-type result

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Abstract. Consider irrational affine subspace $A \subset \mathbb{R}^d$ of dimension a . We prove that the set

$$\{\xi = (\xi_1, \dots, \xi_d) \in A : q^{1/a} \cdot \max_{1 \leq i \leq d} \|q\xi_i\| \rightarrow \infty, \quad q \rightarrow \infty\}$$

is an α -winning set for every $\alpha \in (0, 1/2]$

This simple short communication may be considered as a supplement to our short paper [9].

1. Jarník's result in simultaneous Diophantine approximations. All numbers in this paper are real. Notation $\|\cdot\|$ stands for the distance to the nearest integer. In 1938 V. Jarník (see [1], Satz 9 and [2], Statement **E**) proved the following result.

Theorem 1. (V. Jarník) *Suppose that among numbers ξ_1, \dots, ξ_d there are at least two numbers which are linearly independent over \mathbb{Z} , together with 1. Then*

$$\limsup_{t \rightarrow +\infty} \left(t \cdot \min_{q \in \mathbb{Z}, 1 \leq q \leq t} \max_{1 \leq i \leq d} \|q\xi_i\| \right) = +\infty.$$

Nothing more can be said in a general situation. In 1926 A. Khintchine [3] proved the following result.

Theorem 2. (A. Khintchine) *Let $\psi(t)$ increases to infinity as $t \rightarrow +\infty$. Then there exist two algebraically independent real numbers ξ_1, ξ_2 such that for all t large enough one has*

$$t \cdot \min_{q \in \mathbb{Z}, 1 \leq q \leq t} \max_{i=1,2} \|q\xi_i\| \leq \psi(t).$$

A general form of such a result one can find in Jarník's paper [2]. A corresponding lim sup result is due to J. Lesca [6]:

Theorem 3. (J. Lesca) *Let $d \geq 2$. Let $\psi(t)$ be a positive continuous function in t such that the function $t \mapsto \psi(t)/t$ is a decreasing function. Suppose that*

$$\limsup_{t \rightarrow \infty} \psi(t) = +\infty.$$

Then the set of all vectors $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, containing of algebraically independent elements, such that

$$t \cdot \min_{q \in \mathbb{Z}, 1 \leq q \leq t} \max_{1 \leq i \leq d} \|q\xi_i\| \leq \psi(t)$$

for all t large enough, being intersected with a given open set $\mathcal{G} \subset \mathbb{R}^d$ is of cardinality continuum.

We would like to note that Jarník's Theorem 1 as well as some other theorems by Khintchine and V. Jarník were discussed and generalized in author's survey [7]. In particular in [7], Section 4.1 (see also [8]) one can find an improvement of Theorem 1 in terms of the best approximation vectors.

2. Affine subspaces. Let \mathbb{R}^d be a Euclidean space with the coordinates (x_1, \dots, x_d) , let \mathbb{R}^{d+1} be a Euclidean space with the coordinates (x_0, x_1, \dots, x_d) . Consider an affine subspace $A \subset \mathbb{R}^d$. Let $a = \dim A \geq 1$. Define the affine subspace $\mathcal{A} \subset \mathbb{R}^{d+1}$ in the following way:

$$\mathcal{A} = \{\mathbf{x} = (1, x_1, \dots, x_d) : (x_1, \dots, x_d) \in A\}.$$

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We define *linear* subspace $\mathfrak{A} = \text{span } \mathcal{A}$, as the smallest linear subspace in \mathbb{R}^{d+1} containing \mathcal{A} . So $\dim \mathfrak{A} = a + 1$.

Consider a sublattice $\Gamma(A) = \mathfrak{A} \cap \mathbb{Z}^{d+1}$ of the integer lattice \mathbb{Z}^{d+1} . We see that

$$0 \leq \dim \Gamma(A) \leq a + 1.$$

Of course here for a lattice $\Gamma \subset \mathbb{Z}^{d+1}$ by $\dim \Gamma$ we mean the dimension of the linear subspace $\text{span } \Gamma$.

In the case $\dim \Gamma(A) = a + 1 = \dim \mathfrak{A}$ we define A to be a *completely rational* affine subspace in \mathbb{R}^d . For a completely rational affine subspace A by $d(A)$ we denote the fundamental $(a + 1)$ -dimensional volume of the lattice $\Gamma(A)$.

We see from Dirichlet principle that for any completely rational affine subspace A of dimension a there exists a positive constant $\gamma = \gamma(A)$ such that for any $\xi = (\xi_1, \dots, \xi_d) \in A$ the inequality

$$\max_{1 \leq i \leq d} \|q\xi_i\| \leq \frac{\gamma}{q^{1/a}}$$

has infinitely many solutions in positive integers q .

One can easily see that for any affine subspace A of dimension a the set

$$\Omega = \{\xi = (\xi_1, \dots, \xi_d) \in A : \inf_{q \in \mathbb{Z}_+} q^{1/a} \cdot \max_{1 \leq i \leq d} \|q\xi_i\| > 0\}$$

is an $1/2$ -winning set in A . Here we do not want to discuss the definitions (α, β) -games and (α, β) -winning set or α -winning set. This definitions were given in W.M. Schmidt's paper [10]. All the definitions and basic properties of winning sets one can find in the book [11], Chapter 3. In particular, every α -winning set in A has full Hausdorff dimension. A countable intersection of α -winning sets in A is also an α -winning set.

In the case when A is not a completely rational subspace the result about winning property of the set Ω admits a small improvement. This improvement is related to Jarník's result cited behind.

Theorem 4. *Let*

$$0 < \alpha < 1, \quad 0 < \beta < 1, \quad \gamma = 1 + \alpha\beta - 2\alpha > 0.$$

Suppose that $\dim \Gamma(A) < a$. Then the set

$$\Omega^* = \{\xi = (\xi_1, \dots, \xi_d) \in A : q^{1/a} \cdot \max_{1 \leq i \leq d} \|q\xi_i\| \rightarrow \infty, \quad q \rightarrow \infty\} \quad (1)$$

is (α, β) -winning set in A . In particular, it is an α -winning set for every $\alpha \in (0, 1/2]$.

Here we should note that certain results concerning badly approximable vectors in affine subspaces one can find in [4, 5, 9, 12].

3. Lemmata. Consider the set of all $(a + 1)$ -dimensional complete sublattices of the integer lattice \mathbb{Z}^{d+1} . It is a countable set. One can easily see that for any positive H there exist not more than a finite number of such sublattices Γ with the fundamental volume $\det \Gamma \leq H$. Hence we can order the set $\{V_\nu\}_{\nu=1}^\infty$ of all a -dimensional affine subspaces in \mathbb{R}^d in such a way that values $d_\nu = d(V_\nu) = \det \Gamma(V_\nu)$ form an increasing sequence:

$$1 = d_1 \leq d_2 \leq \dots \leq d_\nu \leq d_{\nu+1} \leq \dots$$

We see that

$$d_\nu \rightarrow \infty, \quad \nu \rightarrow \infty. \quad (2)$$

Some of consecutive values of d_ν may be equal. We define a sequence d_{ν_k} of all different elements from the sequence $\{d_\nu\}$:

$$1 = d_{\nu_1} = \dots = d_{\nu_2-1} < d_{\nu_2} = \dots = d_{\nu_3-1} < d_{\nu_3} = \dots < d_{\nu_k} = d_{\nu_{k+1}} = \dots = d_{\nu_{k+1}-1} < d_{\nu_{k+1}} = \dots$$

(of course $\nu_1 = 1$). For V_j we define the affine subspace $\mathcal{V}_j \subset \mathbb{R}^{d+1}$ as

$$\mathcal{V}_j = \{\mathbf{x} = (1, x_1, \dots, x_d) : (x_1, \dots, x_d) \in V_j\}$$

and consider linear subspace $\mathfrak{V}_j = \text{span } \mathcal{V}_j$.

In the sequel for $\xi = (x_1, \dots, x_d) \in A$ we consider a -dimensional ball

$$B(\xi, \rho) = \{\xi' = (\xi'_1, \dots, \xi'_d) \in A : \max_{1 \leq i \leq d} |\xi_i - \xi'_i| \leq \rho\}$$

and d -dimensional ball

$$\hat{B}(\xi, \rho) = \{\xi' = (\xi'_1, \dots, \xi'_d) \in \mathbb{R}^d : \max_{1 \leq i \leq d} |\xi_i - \xi'_i| \leq \rho\}.$$

Obviously

$$B(\xi, \rho) = \hat{B}(\xi, \rho) \cap A.$$

Lemma 1. *Suppose that $U, V \subset \mathbb{R}^d$ are two affine subspaces. Put $L = U \cap V$ and suppose that $\dim U > \dim L$. Suppose that affine subspace $L' \subset U$ has dimension $\dim L' = \dim U - 1$, and $L' \cap L = \emptyset$. Define $\overline{U} \subset U$ to be a half-subspace with the boundary L' and such that $\overline{U} \cap L = \emptyset$. Then $\text{dist}(\overline{U}, V) > 0$.*

Proof. In affine subspace $\text{aff}(U \cup V)$ of dimension $w = \dim U + \dim V - \dim L$ there exists an affine subspace $L'' \supset L'$ with dimension $\dim L'' = w - 1$ such that $L'' \cap V = \emptyset$. So $\text{dist}(L'', U) > 0$. The subspace L'' divides $\text{aff}(U \cup V)$ into two parts, and lemma follows. \square

Corollary. *Consider two affine subspaces $A, V \subset \mathbb{R}^d$. Suppose that for $\xi \in A$ the ball $B(\xi, \rho) \subset A$ satisfies the property*

$$\text{dist}(B(\xi, \rho), A \cap V) \geq \varepsilon > 0.$$

Then there exists positive $\delta = \delta(A, V, \xi, \varepsilon)$ such that for any $\xi' \in B(\xi, \rho)$ one has

$$\hat{B}(\xi', \delta) \cap V = \emptyset.$$

Proof. From the conditions of our Corollary we see that $\dim(A \cap V) < \dim A$. So we can take a subspace L' of dimension $\dim L' = \dim A - 1$ which separates the ball $B(\xi, \rho)$ from the subspace $A \cap V$ in A . Now we use Lemma 1. \square

Lemma 2. *Let $\rho > 0$ and $\xi \in A$. Consider a ball $\hat{B}(\xi, \rho) \subset \mathbb{R}^d$ such that*

$$\hat{B}(\xi, \rho) \cap \mathfrak{V}_j = \emptyset, \quad 1 \leq j \leq n. \quad (3)$$

Define $k = k(n)$ from the condition

$$\nu_k \leq n < \nu_{k+1}. \quad (4)$$

Put

$$\kappa = \kappa_{d, \xi} = (2\sqrt{d})^a \times \sqrt{1 + (|\xi_1| + 1)^2 + \dots + (|\xi_d| + 1)^2}, \quad \sigma = \sigma_{a, d, \xi} = \frac{1}{\kappa_{d, \xi}(a + 1)!} \quad (5)$$

and

$$T = (\sigma d_{\nu_n} \rho^{-a})^{\frac{1}{a+1}}.$$

Then the set of all rational points $\left(\frac{b_1}{q}, \dots, \frac{b_d}{q}\right) \in \hat{B}(\xi, \rho)$ with $q \leq T$ lie in a certain $(a-1)$ -dimensional affine subspace.

Proof. We may suppose that the set of rational points from $\hat{B}(\xi, \rho)$ with $q \leq T$ consists of more than a points (otherwise there is nothing to prove). We take arbitrary $a + 1$ points

$$\left(\frac{b_{1,j}}{q_j}, \dots, \frac{b_{d,j}}{q_j}\right) \in \hat{B}(\xi, \rho), \quad 1 \leq j \leq a + 1, \quad \text{g.c.d.}(q_j, b_{1,j}, \dots, b_{d,j}) = 1, \quad 1 \leq j \leq a + 1$$

and prove that primitive integer vectors

$$\mathbf{b}_j = (q_j, b_{1,j}, \dots, b_{d,j}), \quad 1 \leq j \leq a + 1 \quad (6)$$

are linearly dependent. Then the lemma will be proved.

All integer vectors (6) belong to the cylinder

$$C = C_\xi(T, \rho) = \{\mathbf{x} = (x_0, x_1, \dots, x_d) \in \mathbb{R}^{d+1} : 0 \leq x_0 \leq T, \max_{1 \leq j \leq d} |x_0 \xi_j - x_j| \leq \rho T\}$$

Suppose that they are linearly independent. Then $\mathfrak{L} = \text{span}(\mathbf{b}_1, \dots, \mathbf{b}_{a+1})$ is an $(a + 1)$ -dimensional completely rational linear subspace. By D we denote the fundamental $(a + 1)$ -dimensional volume of the lattice $\mathfrak{L} \cap \mathbb{Z}^{d+1}$. From (3) we see that

$$\mathfrak{L} \neq \mathfrak{V}_j, \quad 1 \leq j \leq n.$$

From (4) we see that

$$D \geq d_{\nu_n}. \quad (7)$$

Now we consider the section $\mathfrak{L} \cap C$ which is an $(a + 1)$ -dimensional convex polytope. As it is inside C , its $(a + 1)$ -dimensional measure is less than

$$(2\sqrt{d}\rho T)^a \times T \sqrt{1 + (|\xi_1| + 1)^2 + \dots + (|\xi_d| + 1)^2} = \kappa \rho^a T^{a+1} = \kappa \sigma d_{\nu_n}.$$

But the section $\mathfrak{L} \cap C$ consist of $a + 1$ independent points from the lattice $\mathfrak{L} \cap \mathbb{Z}^{d+1}$. For the fundamental volume of this lattice we have lower bound (7). That is why

$$\frac{d_{\nu_n}}{(a + 1)!} \leq \frac{D}{(a + 1)!} < \kappa \sigma d_{\nu_n} = \frac{d_{\nu_n}}{(a + 1)!}.$$

This is a contradiction. Lemma is proved. \square

Lemma 3. (W.M. Schmidt's escaping lemma, Lemma 1B, [11], Chapter 3) *Let t be such that*

$$(\alpha\beta)^t < \frac{\gamma}{2}.$$

Suppose a ball $B_j \subset A$ with the radius ρ_j occurs in the game (as a Black ball). Suppose V is an $(d - 1)$ -dimensional affine subspace passing through the center of the ball B_j . Then White can play in such a way that the ball B_{j+t} is contained in the halfspace Π such that the boundary of Π is parallel to the subspace V and the distance between Π and V is equal to $\frac{\rho_j \gamma}{2}$.

Corollary. *Suppose a ball $B_j \subset A$ with the radius ρ_j occurs in the game (as a Black ball). Suppose that $V, V' \subset A$ are two proper affine subspaces of A . Then White can play in such a way that the distance from the ball B_{j+2t} to each of subspaces V, V' is greater than $\frac{\rho_{j+2t} \gamma}{2}$ (here ρ_{j+2t} is the radius of the ball B_{j+2t}).*

4. Proof of Theorem 4. Suppose that $t = t(\alpha, \beta)$ satisfies the condition of Lemma 3. Put $j_k = 2tk$ and $R_0 = 1$. Suppose that the first Black ball $B_0 \subset A$ with the radius ρ_0 lies inside the

box $\{\xi \in \mathbb{R}^d : \max_{1 \leq i \leq d} |\xi_i| \leq W\}$. We shall prove that White can play in such a way that for any $\xi \in B_{j_r}$ one has

$$\max_{1 \leq i \leq d} \|q\xi_i\| \geq \frac{(\alpha\beta)^t \gamma \rho_0}{2} R^{-\frac{(a+1)r}{a}} \cdot q, \quad \forall q < R_r \quad (8)$$

with a certain R_r which we define later in the inductive step.

We shall prove it by induction in r .

The base of induction is obvious.

Suppose that the ball $B_{j_{r-1}} = B(\xi_{j_{r-1}}, \rho_{j_{r-1}}) \in A$, $\xi_{j_{r-1}} = (\xi_{j_{r-1},1}, \dots, \xi_{j_{r-1},d})$ which occurs as a Black ball satisfies the condition specified. Note that $\rho_{j_{r-1}} = \rho_0(\alpha\beta)^{j_{r-1}}$. Consider the ball $\hat{B}_{j_{r-1}} = \hat{B}(\xi_{j_{r-1}}, 2\rho_{j_{r-1}}) \in \mathbb{R}^d$. Define k_r as the maximal k such that $\hat{B}_{j_{r-1}} \cap \mathfrak{V}_j = \emptyset, 1 \leq j \leq \nu_k$. Then we apply Lemma 2 to see that all rational points $(\frac{b_1}{q}, \dots, \frac{b_d}{q}) \in \hat{B}_{j_{r-1}}$ with

$$q \leq \left(\sigma_{a,d,\xi_{j_{r-1}}} (2\rho_0)^{-a} \right)^{\frac{1}{a+1}} \left(\frac{1}{\alpha\beta} \right)^{\frac{2at(r-1)}{a+1}} d_{\nu_{k_{r-1}}}^{\frac{1}{a+1}}$$

lie in a certain $(a-1)$ -dimensional affine subspace. We denote this subspace by V'_r . As $\max_{1 \leq i \leq d} |\xi_{j_{r-1},i}| \leq W$ we see that

$$\sigma_{a,d,\xi_{j_{r-1}}} \geq \Sigma_{a,d,W} = \frac{1}{(2\sqrt{d})^a \sqrt{1 + (W+1)^2 d} (a+1)!}.$$

We put

$$R_r = (\Sigma_{a,d,W} (2\rho_0)^{-a})^{\frac{1}{a+1}} \left(\frac{1}{\alpha\beta} \right)^{\frac{2at(r-1)}{a+1}} d_{\nu_{k_{r-1}}}^{\frac{1}{a+1}}. \quad (9)$$

By Corollary to Lemma 3 White can play in such a way that

$$\text{dist}(B_{j_r}, V_r) \geq \frac{\gamma \rho_{j_{r-1}+t}}{2} \quad (10)$$

and

$$\text{dist}(B_{j_r}, V'_r) \geq \frac{\gamma \rho_{j_{r-1}+t}}{2} \quad (11)$$

So the inductive step is described and we must show that (8) is valid. But it is clear from (11) that for any $\xi \in B_{j_r}$ one has

$$\max_{1 \leq i \leq d} \|q\xi_i\| \geq \frac{1}{2} \gamma \rho_{j_{r-1}+t} q = \frac{\gamma \rho_0}{2} (\alpha\beta)^{(2t-1)r} q, \quad \forall q < R_r. \quad (12)$$

Moreover by Corollary to Lemma 1 from (10) we see that

$$k_r \rightarrow +\infty, \quad r \rightarrow +\infty.$$

Hence

$$d_{\nu_{k_r}} \rightarrow +\infty, \quad r \rightarrow +\infty. \quad (13)$$

Consider a point $\xi \in \bigcap_j B_j$. For positive integer q define r from the condition

$$R_{r-1} \leq q < R_r.$$

Then we make use of $\xi \in B_{j_r}$. From the inequality $q \geq R_{r-1}$ and (9) we see that

$$\alpha\beta \geq \omega_1 q^{-\frac{a+1}{2atr}} d_{\nu_{k_{r-2}}}^{\frac{1}{2atr}},$$

where $\omega_1 = \omega_1(a, d, W, \alpha, \beta, t) > 0$. We substitute this estimate into (12) to see that

$$\max_{1 \leq i \leq d} \|q\xi_i\| \geq \omega_2 q^{-1/a} d_{\nu_{k_r-2}}^{1/a}, \quad R_{r-1} \leq q < R_r,$$

with positive $\omega_2 = \omega_2(a, d, W, \alpha, \beta, t)$. From (13) for $\xi \in \bigcap_j B_j$ we deduce that

$$q^{1/a} \cdot \max_{1 \leq i \leq d} \|q\xi_i\| \rightarrow +\infty, \quad q \rightarrow \infty.$$

So White can enforce Black to reach a point ξ with the desired properties. Theorem 4 is proved. \square

References

- [1] V. Jarník, Zum Khintchineschen “Übertragungssatz”, Travaux de L’Institut Mathematique de Tbilissi, T 3 (1938), 193 - 212.
- [2] V. Jarník, Eine Bemerkung über diophantische approximationen, Math. Zeitschr. 72, 187 - 191 (1959).
- [3] A.Y. Khinchine, Über eine klasse linear Diophantine Approximationen, Rendiconti Circ. Math. Palermo, 1926, 50, p.170 - 195.
- [4] D. Kleinbock, Extremal supspaces and their submanifolds, GAFA 13:2 (2003), 437 - 466.
- [5] D. Kleinbock, An extension of quantitative nondivergence and applications to Diophantine exponents, Trans. Amer. Math. Soc. 360 (2008), 6497-6523.
- [6] J. Lesca, Sur un résultat de Jarník, Acta Arith. 11 (1966), 359–364.
- [7] N.G. Moshchevitin, Khintchine’s singular Diophantine systems and their applications, Russian Mathematical Surveys. 65:3 43 - 126 (2010).
- [8] N.G. Moshchevitin, Best simultaneous approximations: Norms, signatures, and asymptotic directions, Math. Notes 67:5 (2000), 618–624.
- [9] N. Moshchevitin, On Kleinbock’s Diophantine result, submitted to Proc. Math. Debrecen, preprint available at arXiv:0906.1541 (2009, 2011).
- [10] W.M. Schmidt, On badly approximable numbers and certain games, Trans. Amer. Math. Soc., 623 (1966), p. 178 – 199.
- [11] W.M. Schmidt, Diophantine Approximations, Lect. Not. Math., 785 (1980).
- [12] Y. Zhang, Diophantine exponents of affine subspaces: The simultaneous approximation case, J. Number Theory, 129:8 (2009), 1976-1989.